Reflections on Finite Model Theory

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Traditional Focus of Mathematical Logic

- Study of logics on the class of *all* structures
  - *Gödel's Completeness Theorem*
    is about truth in first-order logic on
    the class of *all* structures (finite & infinite).

- Study of logics on a *fixed infinite* structure
  - *Gödel's Incompleteness Theorem*
    is about truth in first-order logic on
    the structure $\mathcal{N} = (N, +, \times)$ of the integers.
  - *Tarski's Theorem*
    is about truth in first-order logic on
    the structure $\mathcal{R} = (R, +, \times)$ of the reals.

Finite Model Theory

It is the study of logics on classes of finite structures

Examples of Classes

- all finite graphs $G = (V, E)$
- all *ordered* finite graphs $G = (V, E, <)$, where $<$ is a *total order* on V
- restricted classes of finite structures:

  all planar graphs, all trees, ...

Examples of Logics

- First-Order Logic
- Extensions of First-Order Logic:
  - Fragments of Second-Order Logic
  - Logics with Fixed-point Operators
  - Logics with Generalized Quantifiers

Brief History & Overview of the Talk

- Late 1940s to 1970:

  Early scattered results and problems about first-order logic on finite structures.

- Early 1970s to present:
  - Steady development of finite model theory in its own right
  - Extensive interaction with computational complexity, database theory, and asymptotic combinatorics.

- Aims of this talk:
  - Trace some of the early origins
  - Highlight some of the main results
  - Conclude with selected open problems.

Early Origins: Part I

Theorem: Trahtenbrot – 1950

Finite validities *cannot* be axiomatized:

The set of *finitely valid* first-order sentences is *not* recursively enumerable. $\blacksquare$

Striking difference with

Completeness Theorem: Gödel – 1930

Validities can be axiomatized:

The set of *valid* first-order sentences is recursively enumerable. $\blacksquare$

Early Origins: Part II

Theorem: Tarski – 1948

Preservation under substructures theorem

Assume that φ is a first-order sentence such that if φ is true on some structure $\mathbf{A}$, then φ is true on every substructure of $\mathbf{A}$.

There is a universal first-order ψ that is equivalent to φ on all structures.

Conjecture: Scott & Suppes – 1958

Preservation under substructures theorem holds in the finite:

Assume that φ is a first-order sentence such that if φ is true on some finite structure $\mathbf{A}$, then φ is true on every substructure of $\mathbf{A}$.

There is a universal first-order ψ that is equivalent to φ on finite structures.

Early Origins: Part II

Conjecture: Scott & Suppes – 1958

Preservation under substructures theorem holds in the finite.

Theorem: Tait – 1959

The Scott & Suppes conjecture is false:

There is an existential-universal sentence preserved under substructures in the finite, but *not* equivalent to any universal sentence in the finite. ∎

(Rediscovered in the 1980s by Gurevich & Shelah).

Early Origins: Part III

Definition:

The *spectrum* of a first-order sentence ψ is

Spec(ψ) = {m: ψ has a finite model with m elements}

Examples:

- The set of all even numbers

  ψ asserts that "f is a permutation of order 2".

- The set of all powers of prime numbers

  ψ is the conjunction of the field axioms.

The Spectrum Problem:

- Characterize all spectra

  Scholz – 1952

- Are spectra closed under complements?

  Asser – 1956

Research Areas in Finite Model Theory

Descriptive Complexity Theory

Connections between *computational complexity* and *uniform definability* in logics on finite structures.

Expressive Power of Logics

What *can* and what *cannot* be expressed in various logics on classes of finite structures.

Logic & Asymptotic Probabilities

0-1 Laws and *Convergence Laws* for various logics on classes of finite structures.

Computational Complexity

Computational Complexity:

- *The quantitative study of solvability*

  Y. Hartmanis – 1993

- Solvable decision problems are placed into classes according to the time/space resources required to solve them.

Major Complexity Classes:

P	polynomial time
NP	nondeterministic polynomial time
PSPACE	polynomial space

Fact: P $\subseteq$ NP $\subseteq$ PSPACE

Open Problem: Are these containments proper?

The Expressive Power of First-Order Logic

Theorem: Gödel – 1931

First-order logic FO has extremely high power on $\mathbf{N} = (N, +, \times)$.

Every recursive predicate is FO-definable on $\mathbf{N}$.

Fact:

First-order logic FO has severely limited expressive power on the class $\mathcal{G}$ of all finite graphs.

EVEN CARDINALITY and CONNECTIVITY are *not* FO-definable on $\mathcal{G}$.

Fact: On the class of all finite graphs

$$\mathrm{FO} \subsetneq \mathrm{P}.$$

Second-Order Logic

Syntax: FO + Second-Order Quantifiers $\exists R, \forall R$

Semantics: Second-order quantifiers range over sets of nodes, or sets of pairs of nodes, or sets of triples of nodes, and so on.

Simplest Fragments of Second-Order Logic

- Existential Second-Order Logic ESO: Σ^1_1

  $$(\exists R_1)\cdots(\exists R_k)\varphi(R_1,\ldots,R_k),$$

  where $\varphi(R_1,\ldots,R_k)$ is first-order.

- Universal Second-Order Logic USO: Π^1_1

  $$(\forall R_1)\cdots(\forall R_k)\varphi(R_1,\ldots,R_k),$$

  where $\varphi(R_1,\ldots,R_k)$ is first-order.

Examples of ESO-Formulas

- 2-COLORABILITY

$$(\exists P)(\forall x)(\forall y)(E(x,y) \rightarrow (P(x) \leftrightarrow \neg P(y))$$

- 3-COLORABILITY

$$\begin{aligned}
&(\exists P)(\exists Q)(\forall x)(\forall y)[(\neg(P(x) \wedge Q(x))\wedge \\
&\quad (E(x,y) \quad \rightarrow \quad (P(x) \rightarrow \neg P(y))\wedge \\
&\qquad\qquad\qquad\qquad (Q(x) \rightarrow \neg Q(y))\wedge \\
&\qquad\qquad\qquad\qquad (\neg(P(x) \vee Q(x)) \rightarrow (P(y) \vee Q(y)))].
\end{aligned}$$

- HAMILTONIAN PATH

$(\exists R)(R$ is a linear order $\wedge$

$$(\forall x)(\forall y)(\text{“}y = x + 1\text{”} \rightarrow E(x,y))).$$

Descriptive Complexity

Theorem: Fagin – 1972

The following are equivalent for a class $\mathcal{C}$ of finite graphs closed under isomorphisms:

- $\mathcal{C}$ is in NP.
- $\mathcal{C}$ is definable by an ESO-formula. ∎

Remarks:

- Prototypical result of descriptive complexity.
- Machine-independent characterization of NP.
- Reinforces the unity of logic and computation.

NP-Completeness

Fact: 3-COLORABILITY and HAMILTONIAN PATH are NP-complete problems, i.e., every problem in NP can be reduced to them via a polynomial-time reduction.

Problem: Is NP closed under complements?

(if not, then P $\neq$ NP.)

Corollary: The following are equivalent:

- NP is closed under complements.
- ESO on finite structures is closed under complements.
- 3-COLORABILITY is USO-definable (Π^1_1).
- HAMILTONIAN PATH is USO-definable (Π^1_1). ∎

Descriptive Complexity

Question: Fagin's Theorem asserts that NP can be characterized using logic.

Can P and PSPACE be also characterized using logic?

Answer: A qualified "yes".

Both P and PSPACE can be characterized using logics with suitable *fixed-point operators* on classes of *ordered* finite structures

$$\mathbf{A} = (A, <, R_1, \ldots, R_m),$$

where $<$ is a linear order on A.

Logics with Fixed-Point Operators

Motivation: Add recursion to first-order logic by augmenting it with *fixed-point* operators

Formulas as operators:

Let $\varphi(x_1, \ldots, x_k, S)$ be first-order over $\sigma \cup \{S\}$, where S is a k-ary relation symbol not in σ.

- On every $\mathbf{A}$ over σ, it gives rise to an operator

$$\Phi : \mathcal{P}(A^k) \to \mathcal{P}(A^k),$$

where

$$\Phi(P) = \{(a_1, \ldots, a_k) : \mathbf{A} \models \varphi(a_1, \ldots, a_k, P)\}$$

- Iterate Φ : $\quad \varphi^1 = \Phi(\emptyset), \quad \varphi^{m+1} = \Phi(\varphi^m)$

Least Fixed-Point Logic LFP

- φ *positive* in $S \quad \Longrightarrow \quad \Phi$ is *monotone*

$$\varphi^1 \subseteq \varphi^2 \subseteq \ldots \subseteq \varphi^m \subseteq \varphi^{m+1} \subseteq \ldots$$

- Φ has a *least fixed-point* φ^∞

$$\varphi^\infty = \bigvee_m \varphi^m$$

Definition: LFP $=$ FO $+$ Least Fixed-Points

Fact: FO $\subseteq$ LFP $\subseteq$ P

Example: CONNECTIVITY

$$\begin{aligned}
\varphi(x,y,S) &\equiv E(x,y) \lor (\exists z)(E(x,z) \land S(z,y)) \\
\varphi^m(x,y) &\equiv \text{there is a path of length} \\
& \quad\; \text{at most } m \text{ from } x \text{ to } y \\
\varphi^\infty(x,y) &\equiv \text{there is a path from } x \text{ to } y
\end{aligned}$$

$$\mathbf{G} = (V,E) \text{ is connected } \iff \mathbf{G} \models (\forall x)(\forall y)\varphi^\infty(x,y)$$

Corollary: CONNECTIVITY is LFP-definable.

Example: ACYCLICITY

$$\begin{aligned}
\psi(x,S) &\equiv (\forall y)(E(y,x) \rightarrow S(y)) \\
\psi^1(x) &\equiv \text{indegree}(x) = 0 \\
\psi^2(x) &\equiv (\forall y)(E(y,x) \rightarrow \text{indegree}(y) = 0) \\
\vdots \quad & \quad \vdots \\
\psi^\infty(x) &\equiv \text{no path through } x \text{ leads to a cycle}
\end{aligned}$$

$$\mathbf{G} = (V,E) \text{ is acyclic } \iff \mathbf{G} \models (\forall x)\psi^\infty(x)$$

Corollary: ACYCLICITY is LFP-definable.

Example

Example: PATH SYSTEMS $\mathbf{T} = (F, A, R)$

Given a finite set $A \subseteq F$ of *axioms* and a ternary *rule of inference* $R \subseteq F^3$, compute the *theorems* of this system.

$$\begin{aligned} \psi(x, S) &\equiv A(x) \vee (\exists y)(\exists z)(S(y) \wedge S(z) \wedge R(x, y, z)) \\ \psi^\infty(x) &\equiv x \text{ is a theorem of the system } \mathbf{T} \end{aligned}$$

Theorem: Cook – 1974

The least fixed-point ψ^∞ of ψ is P-complete. ∎

Corollary:

LFP can express P-complete queries. ∎

Descriptive Complexity Theory

Fact: On the class $\mathcal{G}$ of all finite graphs

$$\mathrm{FO} \subsetneq \mathrm{LFP} \subsetneq \mathrm{P}.$$

Reason for proper containments:

- CONNECTIVITY is not FO-definable.
- EVEN CARDINALITY in not LFP-definable. ∎

Theorem: Immerman – 1982, Vardi – 1982

On the class $\mathcal{O}$ of all finite ordered graphs $G = (V, <, E)$, it is the case that

$$\mathrm{LFP} = \mathrm{P}. \quad \blacksquare$$

Research Areas in Finite Model Theory

Descriptive Complexity Theory

Connections between *computational complexity* and *uniform definability* in logics on finite structures.

Expressive Power of Logics

What *can* and what *cannot* be expressed in various logics on classes of finite structures.

Logic & Asymptotic Probabilities

0-1 Laws and *Convergence Laws* for various logics on classes of finite structures.

Logic & Asymptotic Probabilities

$\mathcal{G}_n$:	The class of all graphs with n nodes
μ_n :	Probability measure on $\mathcal{G}_n$, $n \geq 1$
$\mathcal{C}$:	A class of graphs (a *property* of graphs)
$\mu_n(\mathcal{C})$:	Probability of $\mathcal{C}$ on $\mathcal{G}_n$ w.r.t. μ_n, $n \geq 1$.

Definition: *Asymptotic probability of class* $\mathcal{C}$

$$\mu(\mathcal{C}) \quad = \quad \lim_{n\to\infty} \mu_n(\mathcal{C}), \quad \text{provided the limit exists}$$

Example: *Uniform Measure*

$\mu_n(\mathcal{C})$: Fraction of graphs with n nodes in $\mathcal{C}$

- $\mu(\text{4-}\textsc{regular}) = 0$
- $\mu(\text{2-}\textsc{Colorable}) = 0$
- $\mu(\textsc{Hamiltonian Path}) = 1$
- $\mu(\textsc{Even Cardinality})$ does *not* exist

0-1 Laws and Convergence Laws

Problem: Is there a connection between the *definability* of a class $\mathcal{C}$ in some logic $\mathcal{L}$ and its *asymptotic probability* w.r.t. a given measure?

Definition: Let $\mathcal{L}$ be a logic

- $\mathcal{L}$ has a *0-1 law* with respect to μ_n, $n \geq 1$, if

$$\mu(\mathcal{C}) = 0 \quad \text{or} \quad \mu(\mathcal{C}) = 1$$

  for every $\mathcal{L}$-definable class $\mathcal{C}$.

- $\mathcal{L}$ has a *convergence law* w.r.t. μ_n, $n \geq 1$, if $\mu(\mathcal{C})$ exists for every $\mathcal{L}$-definable class $\mathcal{C}$.

Problem: Does *first-order* logic have a 0-1 with respect to the *uniform* measure?

0-1 Law for First-Order Logic

Glebskii, Kogan, Liogonki, Talanov – 1969
Fagin – 1972

Theorem: The 0-1 law holds for first-order logic with respect to the uniform measure. ∎

Transfer Theorem: Fagin – 1972

There is a unique countable graph **R** such that for every first-order sentence φ

$$\mu(\varphi) = 1 \quad \text{if and only if} \quad \mathbf{R} \models \varphi. \quad \blacksquare$$

Note:

- **R** is Rado's graph: the unique countable *homogeneous* and *universal* graph
- **R** is characterized by a set of first-order *extension axioms*:
  "every finite extension is possible"

$\boxed{\textbf{Extension Axioms } \sigma_k\textbf{, } k \geq 1}$

$$(\forall x_1) \cdots (\forall x_k)(\forall y_1) \cdots (\forall y_k)[\bigwedge_{i,j}(x_i \neq y_j) \rightarrow$$

$$(\exists w) \bigwedge_{i=1}^{k}(E(w, x_i) \wedge \neg E(w, y_i))]$$

Lemma 1: $T = \{\sigma_k,\ k \geq 1\}$ has a unique countable model (Rado's graph $\mathbf{R}$).

Proof: *back* and *forth* argument.

Lemma 2: For every $k \geq 1$, we have $\mu(\sigma_k) = 1$

Proof: for every $k \geq 1$, there is a θ such that

$$0 < \theta < 1 \text{ and } \mu_n(\neg\sigma_k) \leq \theta^n.$$

Decision Problem for 0-1 Law

Problem:

Given a first-order sentence φ, tell whether

$$\mu(\varphi) = 0 \quad \text{or} \quad \mu(\varphi) = 1.$$

Note:

- By the Transfer Theorem, this is equivalent to deciding first-order truth on $\mathbf{R}$.
- Fagin's proof shows it is solvable.

Theorem: Grandjean – 1983

The decision problem for the 0-1 law for first-order logic is PSPACE complete. ∎

Truth vs. Almost Sure Truth

First-Order Truth

everywhere true (valid)	somewhere true & somewhere false	everywhere false (contradiction)

Almost Sure First-Order Truth

almost surely true	almost surely false

- Testing if a first-order sentence is *true* on all finite graphs is an *unsolvable* problem

- Testing if a first-order sentence is *almost surely true* on all finite graphs is a *solvable* problem; in fact, it is PSPACE-complete.

Three Directions of Research

- First-order 0-1 laws for restricted classes of finite structures under the uniform measure.

- First-order 0-1 laws on graphs under *variable* probability measures.

- 0-1 laws for *extensions* of first-order logic.

Note:

- 0-1 laws are new phenomena that are meaningful only in the context of finite structures.

- Finiteness is a feature, not a limitation.

0-1 Laws for Fragments of ESO

Fact:

- The convergence law fails for ESO:

  EVEN CARDINALITY is ESO-definable.

  Nonetheless,

- Many *natural* NP-complete problems have asymptotic probability 0 or 1
  - 3-COLORABILITY
  - HAMILTONIAN PATH
  - SATISFIABILITY
  - MONOCHROMATIC TRIANGLE
  - KERNEL

Question: Do 0-1 laws hold for *fragments* of ESO?

0-1 Laws for Fragments of ESO

(joint work with Moshe Y. Vardi)

Idea:

Pursue 0-1 laws for fragments of ESO obtained by restricting the first-order part $\varphi(\overline{S})$ of ESO-sentences $(\exists \overline{S})\varphi(\overline{S})$.

Guiding Principle:

Skolem Normal Form for ESO:

$$\underbrace{(\exists \overline{S})}_{\text{SO}}\underbrace{(\exists \overline{x})(\forall \overline{y})(\exists \overline{z})}_{\text{FO}}\underbrace{\theta(\overline{S}, \overline{x}, \overline{y}, \overline{z})}_{\text{quantif.-free}}$$

Thus, consider first-order *prefix* classes Ψ that are subclasses of $\exists^*\forall^*\exists^*$.

0-1 Laws for Fragments of ESO

Theorem: K ... & Vardi

- For every $\Sigma^1_1(\exists^*\forall^*)$-sentence ψ,

$$\mu(\psi) = 1 \quad \text{if and only if} \quad \mathbf{R} \models \psi.$$

- The 0-1 law holds for $\Sigma^1_1(\exists^*\forall^*)$.
- The associated decision problem for the 0-1 law is NEXPTIME-complete. ∎

Theorem: K ... & Vardi

- For every $\Sigma^1_1(\exists^*\forall\exists^*)$-sentence ψ,

$$\mu(\psi) = 1 \quad \text{if and only if} \quad \mathbf{R} \models \psi.$$

- The 0-1 law holds for $\Sigma^1_1(\exists^*\forall\exists^*)$.
- The associated decision problem for the 0-1 law is NEXPTIME-complete. ∎

Theorem: Pacholski - Szwast

The convergence law *fails* for $\Sigma^1_1(\forall\forall\exists)$. ∎

0-1 Laws for Fragments of ESO

Class	0-1 Law	Decision Problem
$\Sigma^1_1(\exists^*\forall^*)$	Yes	NEXPTIME-complete
$\Sigma^1_1(\exists^*\forall\exists^*)$	Yes	NEXPTIME-complete
$\Sigma^1_1(\exists^*\forall\forall\exists^*)$	No	Unsolvable

The above were the key results in establishing the following

Classification Theorem: The Bernays-Schönfinkel class $\exists^*\forall^*$ and the Ackermann class $\exists^*\forall\exists^*$ are the *only* prefix classes Ψ of first-order logic such that the 0-1 law holds for the corresponding fragment $\Sigma^1_1(\Psi)$ of existential second-order logic.

Logics without Equality

Theorem: Gödel – 1932

SAT is decidable for $\forall\forall\exists$ *without* $=$

Theorem: Le Bars – 1998

The convergence law *fails* for $\Sigma^1_1(\forall\forall\exists)$ *without* $=$

Hint of Proof:

A variant of the KERNEL property is expressible in $\Sigma^1_1(\forall\forall\exists)$ *without* $=$, but has no asymptotic probability. ∎

Open Problems: Part I

The Spectrum Problem: Asser – 1956

Are spectra closed under complements?

This problem is still open ...

Theorem: Jones & Selman – 1974

The following are equivalent for a set S of positive integers:

- S is the spectrum of a first-order sentence.
- S is in NEXPTIME. ∎

Corollary: The following are equivalent:

- Spectra are closed under complements.
- NEXPTIME is closed under complements. ∎

Open Problems: Part II

Fact: Most classical preservation theorems fail in the finite.

In particular, both the preservation under extensions theorem and the preservation under onto homomorphisms fail in the finite.

Preservation under Homomorphisms:

Assume that φ is a first-order sentence such that if a finite structure $\mathbf{A}$ satisfies φ and $h : \mathbf{A} \rightarrow \mathbf{B}$ is a homomorphism from $\mathbf{A}$ into a finite structure $\mathbf{B}$, then $\mathbf{B}$ also satisfies φ.

Does there exist an existential positive first-order sentence ψ that is equivalent to φ in the finite?

Open Problems: Part III

Theorem: Immerman – 1982, Vardi – 1982

On ordered finite graphs, P $=$ LFP. ∎

Open Problem: Chandra & Harel – 1982

Is the set of polynomial-time properties of finite graphs recursively enumerable?

Open Problem: Gurevich – 1985

Is there a logic for polynomial time on finite graphs?

Note: Much of the research in the expressive power of logics on finite structures has been directly motivated by these problems.

Open Problems: Part IV

Theorem: Lindström – 1969

First-order logic is the strongest logic possessing the countable compactness theorem and the countable Löwenheim-Skolem theorem. ∎

Note: This result was the catalyst for the development of abstract model theory.

Open Problem:

Are there Lindström-type characterizations of first-order logic and least fixed-point logic in the finite?

Open Problems:

Is there an abstract finite model theory?