Computing the Supremum of the Real Roots of a Parametric Univariate Polynomial

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Plan

Supremum of the Real Roots of a Parametric Polynomial

Input

- Let $W = W_1, \ldots, W_m$ and $H = H_1, \ldots, H_n$ be two (disjoint) sets of variables.
- Let $p \in \mathbb{R}[W, H][X]$ be univariate in X.

Output

- The supremum $x_{\sup}(h)$ of the set

$$\Pi_h = \{x \in \mathbb{R} \mid (\exists (w_1, \ldots, w_m) \in \mathbb{R}^m) \;\; p_{w,h}(x) = 0\} \qquad (1)$$

  where $p_{w,h}$ is the polynomial of $\mathbb{R}[X]$ obtained by evaluating p at $W = w$ and $H = h$, for $h \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$.
- That is, in a more compact form:

$$x_{\sup}(h) = \sup_{w,\, p(w,h,x)=0} x. \qquad (2)$$

Examples

Recall the problem

Compute $x_{\sup}(h) \;=\; \sup_{w,\,p(w,h,x)=0}\; x$.

Example

For $p_1 = h_1 x - w_1$, we have $x_{\sup}(h_1) = +\infty$ for all $h_1 \in \mathbb{R}$.

Example

For $p_2 = h_1^2 x - w_1^2 - 1$, we have $x_{\sup}(h_1) = +\infty$ if $h_1 \neq 0$ and $-\infty$ otherwise.

Example

Finally for $p_3 = x + h_1 w_1^2 - h_1 - 1$, we have

$$x_{\sup}(h_1) = \begin{cases} +\infty & h_1 < 0 \\ h_1 + 1 & h_1 \geq 0. \end{cases}$$

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Compute $x_{\sup}(h) = \sup_{w,\, p(w,h,x)=0} x$.

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Plan

Linear dynamical systems

Notations

- Let A, B, C, D be real matrices with respective formats $n \times n$, $n \times m$, $p \times n$, $p \times m$.
- Consider the linear dynamical system:

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases} \tag{3}$$

- x, y, u are the state vector, output vector and control vector, respectively.
- We assume A *stable*, that is, when all its eigenvalues of A have negative real part.
- The transfer matrix is $G(s) = C(sI_n - A)^{-1}B + D$.

$\mathcal{H}_\infty$ Norm

Definition

The $\mathcal{H}_\infty$ norm of the transfer matrix is

$$||G(s)||_\infty = \sup_{\Re(s)>0} \sigma_{\max}(G(s)) = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(\imath\omega)). \quad (4)$$

Here we have $\sigma_{\max}(F) = \lambda_{\max}^{1/2}(F^* F)$, where $\sigma_{\max}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote respectively the maximum singular value and maximum eigenvalue of a real square matrix.

Comments

- In robust control, $||G(s)||_\infty$ takes the role of a robustness measure.
- In model order reduction, $||G(s)||_\infty$ is used as an error measure.
- Many methods (all numerical) exist and are limited to linear systems free of parameters.
- Among them, (Kanno & Smith, JSC 2006) achieve *validated numerical computation* through univariate polynomial real root isolation.
- Observe that, indeed, $||G(s)||_\infty$ is the supremum (of the square root) of the (real) root of the characterisitic polynomial of $G(\imath\omega)$.

Plan

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- Let $p \in \mathbb{R}[W, H][X]$ be univariate in X.

Output

- The supremum $x_{\sup}(h)$ of the set

$$\Pi_h = \{x \in \mathbb{R} \mid (\exists (w_1, \ldots, w_m) \in \mathbb{R}^m) \;\; p_{w,h}(x) = 0\} \tag{5}$$

where $p_{w,h}$ is the polynomial of $\mathbb{R}[X]$ obtained by evaluating p at $W = w$ and $H = h$, for $h \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$.

- That is, in a more compact form:

$$x_{\sup}(h) = \sup_{w,\, p(w,h,x)=0} x. \tag{6}$$

Intuition

Non-parametric case

- Assume also $m = 1$ and write $W = W_1$, so $p \in \mathbb{R}[W, X]$ is bivariate.
- We view $p \in [X][W]$ as a parametric polynomial with parameter X.
- Intuitively, x_{sup} must have a special relation to $p \in [X][W]$.
- In fact, the theory of the border polynomial suggests that x_{sup} should be one of the roots of the border polynomial of $p \in [X][W]$.

Parametric case

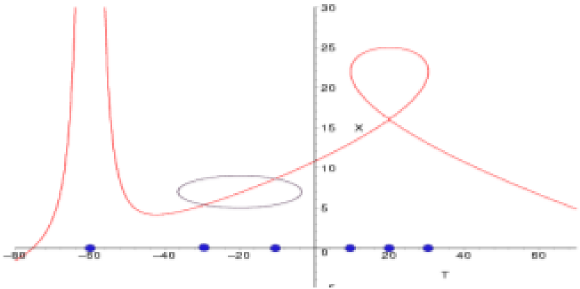
- Assume $m = 1$, $n = 1$ and write $W = W_1$, $H = H_1$ so $p \in \mathbb{R}[H][W, X]$ is bivariate over $\mathbb{R}[H]$.
- Intuitively, computing $x_{\mathrm{sup}}(h)$ should reduce to compute x_{sup} by means of a case discussion.
- In fact, the theory of the border polynomial will support that idea modulo some technical difficulties, which can be
  - handled in theory at some expense or,
  - ignored in practice at no cost.

Plan

Solving parametric polynomial systems

Intuition

- "Generically", the properties of solutions depend on the parameter values continuously. By properties, we mean number, shape.
- "Generically", the points related to "discontinuity" are few.
- In the figure below, we view a polynomial $p(W, X)$ defining a curve $p(W, X) = 0$ as univariate in W over $\mathbb{R}[X]$.



Parametric polynomial systems

Notations

- Let $F, H \subset \mathbb{R}[H, X]$ be two sets of polynomials in variables $X = X_1, \ldots, X_s$ and parameters $H = H_1, \ldots, H_n$.
- The polynomials $f \in F$ and $h \in H$ define the equations and inequations of a parametric polynomial system S.
- We consider the standard projection on the parameter space:

$$\begin{array}{c} \Pi_{\mathrm{H}} : Z(S) \subset \mathbb{C}^{s+n} \mapsto \mathbb{C}^n \\ \Pi_{\mathrm{H}}(x_1, \ldots, x_s, h_1, \ldots, h_n) = (h_1, \ldots, h_n) \end{array} \tag{7}$$

Technical assumption

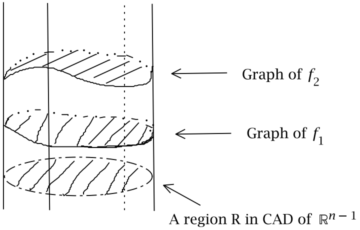
The saturated ideal $\mathcal{I} := \langle F \rangle : (\prod_{h \in H} h)^\infty$ is of dimension n and H is maximally algebraically independent modulo $\mathcal{I}$.

Two notions of continuity

Definition

Let $\alpha \in \mathbb{C}^d$. We say that S is

1. *Z-continuous* at α: if there exists an open ball $\mathcal{O}_\alpha$ centered at α s.t. for any $\beta \in \mathcal{O}_\alpha$ we have $\#(Z(S(\beta)) = \#(Z(S(\alpha))$.
2. Π_{H}*-continuous* at α: if there exists an open ball $\mathcal{O}_\alpha$ centered at α and a finite partition, say $\{C_1, \ldots, C_k\}$ of $\Pi_{\mathrm{H}}^{-1}(\mathcal{O}_\alpha) \cap Z(S)$ such that for each $j \in \{1, \ldots, k\}$ $\Pi_{\mathrm{H}}|_{C_j} : C_j \xrightarrow{\Pi_{\mathrm{H}}} \mathcal{O}_\alpha$ is a diffeomorphism.



Border polynomial and discriminant variety

We formulate these two concepts using the previous *continuity* notions.

Border polynomial (Yang, Xia and Hou, 1999)

A non-zero polynomial b in $\mathbb{Q}[U]$ is called a *border polynomial* (BP) of the parametric polynomial system S if the zero set $V(b)$ of b in $\mathbb{C}^d$ contains all the points at which S is not Z-continuous.

Discriminant variety (Lazard and Rouillier, 2007)

An algebraic set $\mathcal{W} \subsetneq \mathbb{C}^d$ is a *discriminant variety* of the parametric polynomial system S if $\mathcal{W}$ contains all the points at which S is not Π_{H}-continuous.

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Useful properties toward computing

Proposition

Π_{H}-*continuity* implies Z-continuity.

Proposition

If S consists of a single equation $f \in \mathbb{R}[H, X]$, with $X = X_1$, then $\mathrm{discrim}(f, X)\,\mathrm{lcoeff}(f, X)$ is a border polynomial for S.

Proposition

If S consists of a single equation and b is a border polynomial of S then $Z(b)$ is (the minimal) discriminant variety of S.

Corollary

If S consists of a single equation $f \in \mathbb{R}[H, X]$, with $X = X_1$, then the number of X-roots of S is constant above each connected component of the complement of $Z(b)$, with b as above.

See the details in (M^3, B. Xia & R. Xiao, MCS 2012)

Plan

SupRealRoot: non-parametric case

Setting

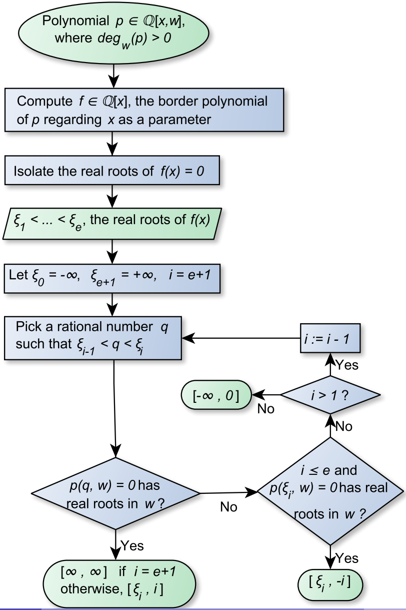
Let $\xi_1 < \cdots < \xi_e$ be the real roots of f, with

$$f = \mathrm{lcoeff}_W(p) \cdot \mathrm{discrim}_W(p), \tag{8}$$

Define $\xi_0 = -\infty$ and $\xi_{e+1} = +\infty$. The algorithm below computes $x_{\sup} = \sup\{x \in \mathbb{R} \mid \exists w \in \mathbb{R} \;\; p(w, x) = 0\}$.

SupRealRoot(p) begin
for $i = e + 1$ downto 1 by -1 do {
let q be a rational number s.t. $\xi_{i-1} < q < \xi_i$
if $p(q, W) = 0$ has real roots in W then return ξ_i
if $i \leq e$ and $p(\xi_i, W) = 0$ has real roots in W then return ξ_i
}
return ξ_0
end

SupRealRoot: non-parametric case



From the non-parametric case to the parametric one

From now on the polynomial

$$f = \mathrm{lcoeff}_W(p) \cdot \mathrm{discrim}_W(p)$$

depends on H and X

New difficulties

1. The zeros of f depend on H: their number may depend on H as well!
2. Moreover, the zero set $Z(f)$ is no longer a finite set of points: it contains higher dimensional components.

Solutions

1. Decompose the H-space into regions above which the zeros of f are given by disjoint (and continuous) graphs.
2. Back-up to a more general tool (based on CAD) when things go wrong.

Plan

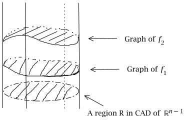
Real comprehensive triangular decomposition (RCTD)

Input

A parametric semi-algebraic system $S \subset \mathbb{Q}[H][X]$.

Output

- A partition of the whole parameter space into connected cells, such that above each cell, the constructible system associated to S
  1. either has infinitely many complex solutions,
  2. or S has no real solutions
  3. or S has finitely many real solutions which are continuous functions of parameters with disjoint graphs
- A description of the solutions of S as functions of parameters by triangular systems in case of finitely many complex solutions.



Example

A RCTD of the system

$$\begin{cases} x(1+y^2)-s=0 \\ y(1+x^2)-s=0 \\ x>0, y>0, s>0 \end{cases}$$

is as follows

1. $s \leq 0, \longrightarrow \{\ \}$
2. $s > 0, s \leq 2 \longrightarrow \{T_1\}$
3. $s > 2 \longrightarrow \{T_1, T_2\}$

where

$$T_1 = \begin{cases} (x^2+1)y-s=0 \\ x^3+x-s=0 \\ x>0 \\ y>0 \end{cases} \qquad T_2 = \begin{cases} xy-1=0 \\ x^2-sx+1=0 \\ x>0 \\ y>0 \end{cases}$$

Plan

ParametricSupRealRoot

Regarding H as parameters, apply RealComprehensiveTriangularize to

$$f = \mathrm{lcoeff}_W(p) \times \mathrm{discrim}_W(p) \ \in \mathbb{R}[H, X].$$

For each cell C_i which is full-dimensional:

(1) Obtain a sample point v_i of the cell C_i

(2) Call the command SupRealRoot at $h = v_i$. Three cases arise.

  (2.1) If SupRealRoot returns a pair of the form $[\xi, m]$ with $\xi \in \{+\infty, -\infty\}$ then the function ParametricMaxRealRoot returns $[\xi, C_i]$.

  (2.2) If SupRealRoot returns a pair of the form $[\xi, m]$ where $m > 0$ holds, then we compute the polynomial g which has ξ as its j-th real root at $h = v_i$ and returns $[[j, g], C_i]$.

  (2.3) In all other cases, apply a CAD-based approach, say computing a CAD of $p(x, w, h) = 0$ for $h < x < w$.

Plan

Example 1

```
> Gs := Matrix([1/((s^2+2*c*s+1)*(s+1))]);
```

$$Gs := \left[\frac{1}{(s^2 + 2\,c\,s + 1)\,(s+1)} \right]$$

```
> Hs := ParametricHinfinityNorm(Gs, 's', [c>0, c<=1]);
```

Hs := [[[[1, *squarefree_semi_algebraic_system*], [*cad_cell*]], [[1, *squarefree_semi_algebraic_system*], [*cad_cell*]], [[1, *squarefree_semi_algebraic_system*], [*cad_cell*]], [[2, *squarefree_semi_algebraic_system*], [*cad_cell*]]], *polynomial_ring*]

```
> Display(Hs[1][1], Hs[-1]);
```

$$[[1, x - 1 = 0], [c = 1]]$$

```
> Display(Hs[1][2], Hs[-1]);
```

$$\left[[1, x - 1 = 0], \left[\mathrm{And}\left(\frac{1}{2} < c, c < 1\right)\right]\right]$$

```
> Display(Hs[1][3], Hs[-1]);
```

$$\left[[1, x - 1 = 0], \left[c = \frac{1}{2}\right]\right]$$

```
> Display(Hs[1][4], Hs[-1]);
```

$$\left[\left[2, (256\,c^8 - 768\,c^6 + 768\,c^4 - 256\,c^2)\,x^2 + (256\,c^6 + 32 - 480\,c^4 + 192\,c^2)\,x - 27 = 0\right], \left[\mathrm{And}\left(0 < c, c < \frac{1}{2}\right)\right]\right]$$

Example 2

```
> A := Matrix([[0,1], [-k/m,-b/m]]): B := Matrix([[0], [1/m]]): C := Matrix([1,0]):
  T := DynamicSystems:-TransferFunction(A,B,C):
  T:-tf;
```

$$\left[\frac{1}{m\, s^2 + b\, s + k} \right]$$

```
> Hm := ParametricHinfinityNorm(T:-tf, 's', [m>0, k>0, b>0]);
```

Hm := [[["Not full-dimension, not processed", [*cad_cell*, *cad_cell*]], [[1, *squarefree_semi_algebraic_system*], [*cad_cell*, *cad_cell*]], [[1, *squarefree_semi_algebraic_system*], [*cad_cell*]]], *polynomial_ring*]

```
> Display(Hm[1][1], Hm[-1]);
```

$$\left[\text{"Not full-dimension, not processed"}, \left[\left[\begin{cases} k = \frac{1}{4}\frac{b^2}{m} \\ 0 < m \\ 0 < b \end{cases}, \begin{cases} k = \frac{1}{2}\frac{b^2}{m} \\ 0 < m \\ 0 < b \end{cases} \right] \right] \right]$$

```
> Display(Hm[1][2], Hm[-1]);
```

$$\left[[1, k^2 x - 1 = 0], \left[\left[\begin{cases} \mathrm{And}\left(0 < k, k < \frac{1}{4}\frac{b^2}{m}\right) \\ 0 < m \\ 0 < b \end{cases}, \begin{cases} \mathrm{And}\left(\frac{1}{4}\frac{b^2}{m} < k, k < \frac{1}{2}\frac{b^2}{m}\right) \\ 0 < m \\ 0 < b \end{cases} \right] \right] \right]$$

```
> Display(Hm[1][3], Hm[-1]);
```

$$\left[[1, (-b^4 + 4\, m\, k\, b^2)\, x - 4\, m^2 = 0], \left[\left[\begin{cases} \frac{1}{2}\frac{b^2}{m} < k \\ 0 < m \\ 0 < b \end{cases} \right] \right] \right]$$

Plan

Concludng remarks

- Taking advantage of the notion of border polynomial and triangular decomposition techniques, we have presented an algorithm and its implementation for computing the supremum of the real roots of a parametric univariate polynomial.
- The precise formulation of this problem (with the bivariate polynomial $p(W, X)$ whose coefficients are real polynomials in H) targets the computation of the $\mathcal{H}_\infty$ norm of the transfer matrix of a linear dynamical system with parametric uncertainty.
- Our implementation allows us to solve the vast majority of the examples that we have found in the literature. A few examples (like the 2-mass-2-spring-2-dampler system) cannot be solved by our code without specializing some of the parameters. However, our preliminary implementation offers several opportunities for optimization. Work is in progress!