Automatic Theorem Proving with Vampire II

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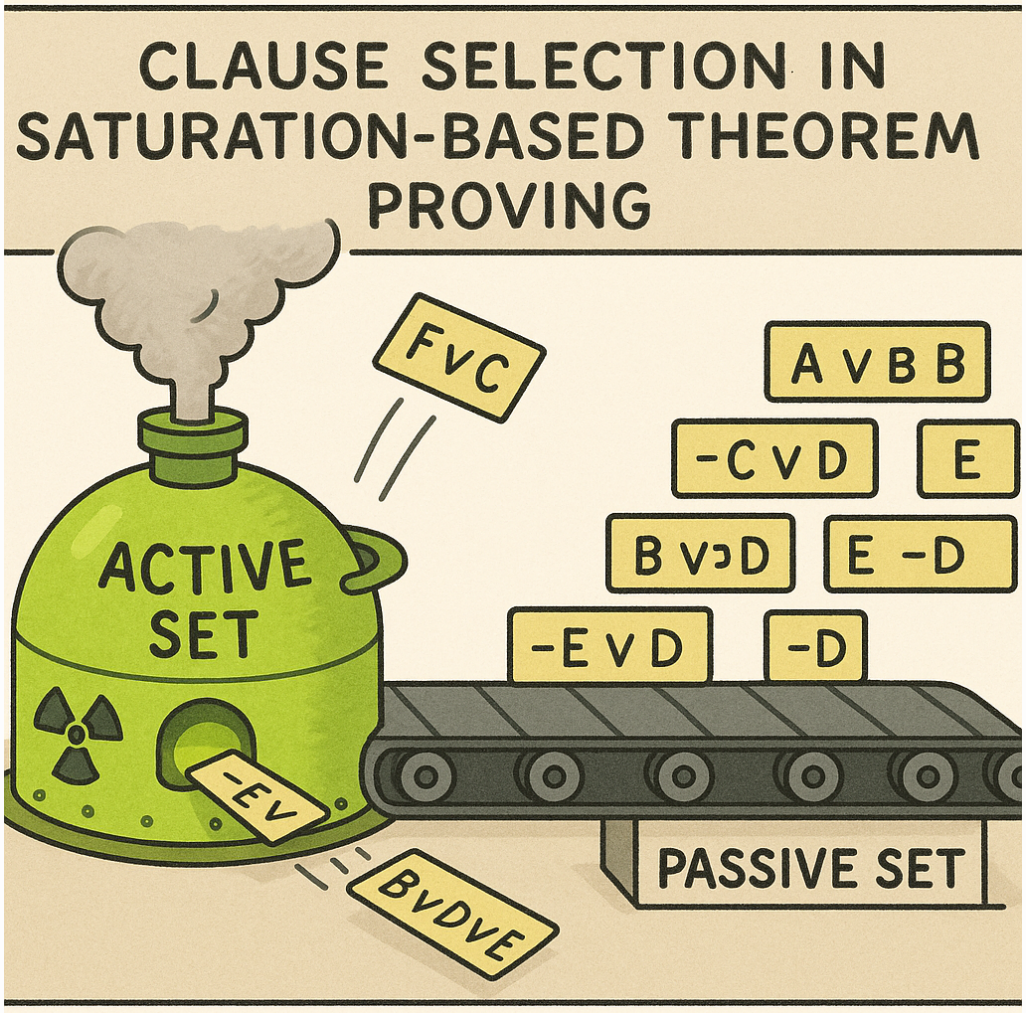
Czech Technical University in Prague, Czech Republic

VTSA 2025, Liege, Belgium

Outline

Saturation-based Theorem Proving



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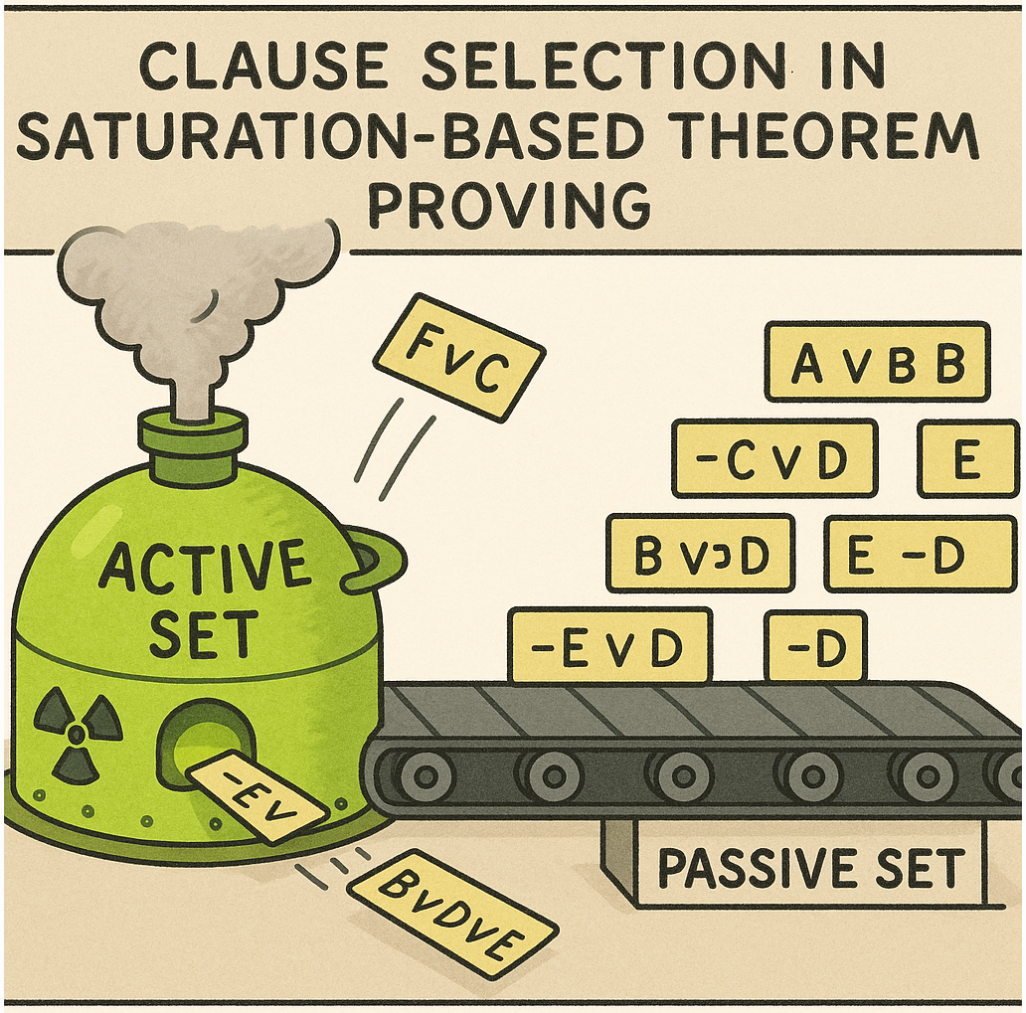
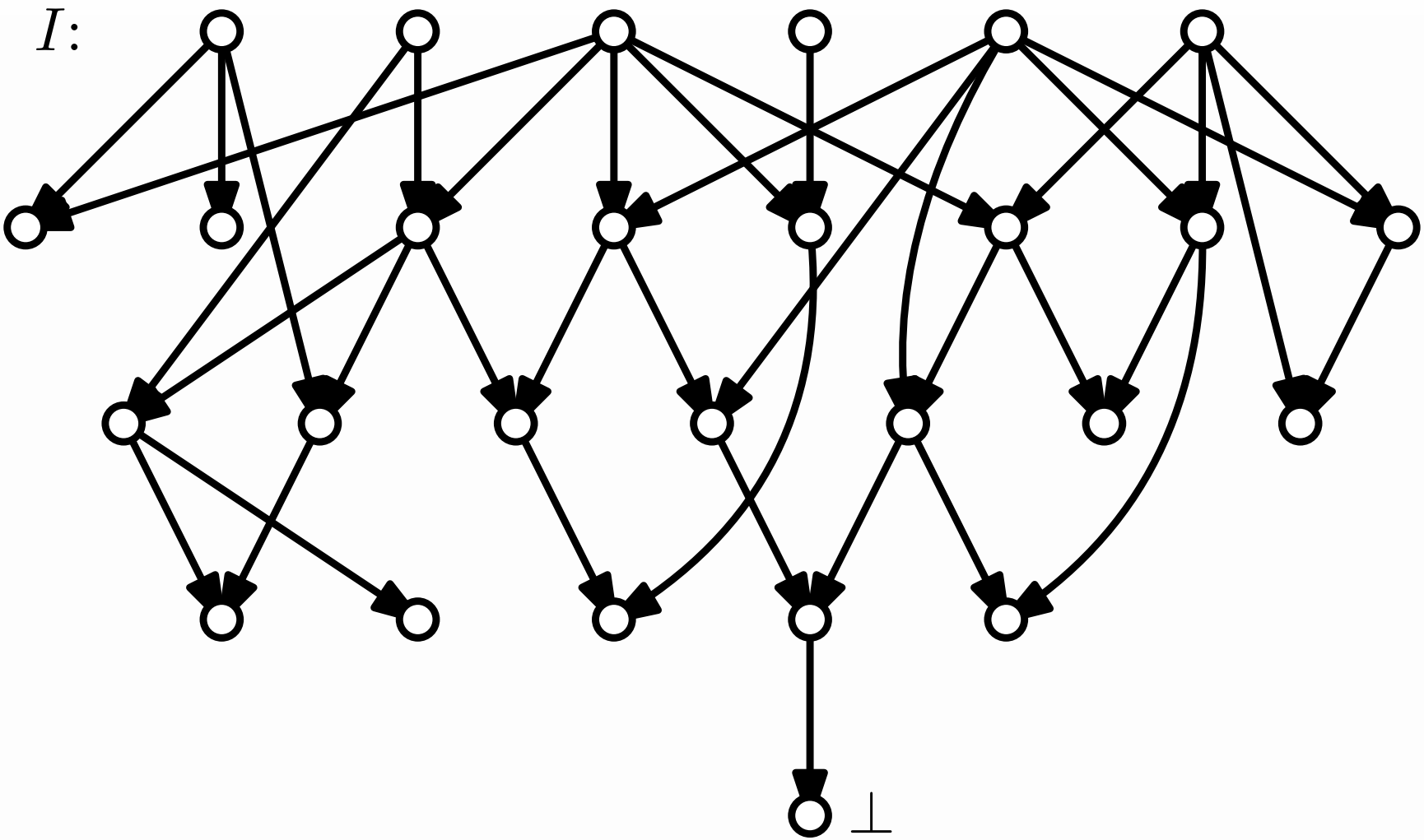


At a typical successful end: $|Passive| \gg |Active| \gg |Proof|$

The Proof Is Often Just A Tiny Part



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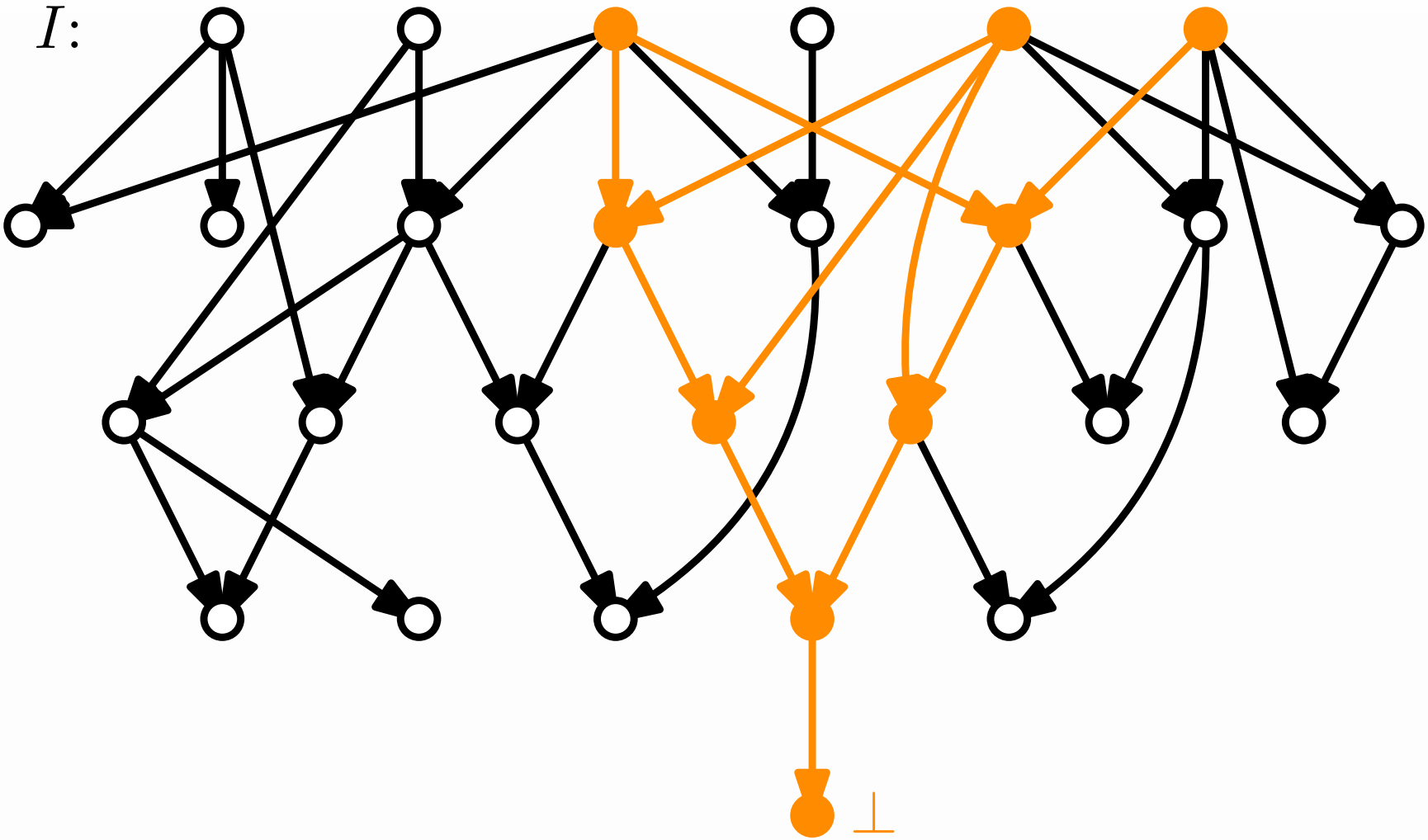

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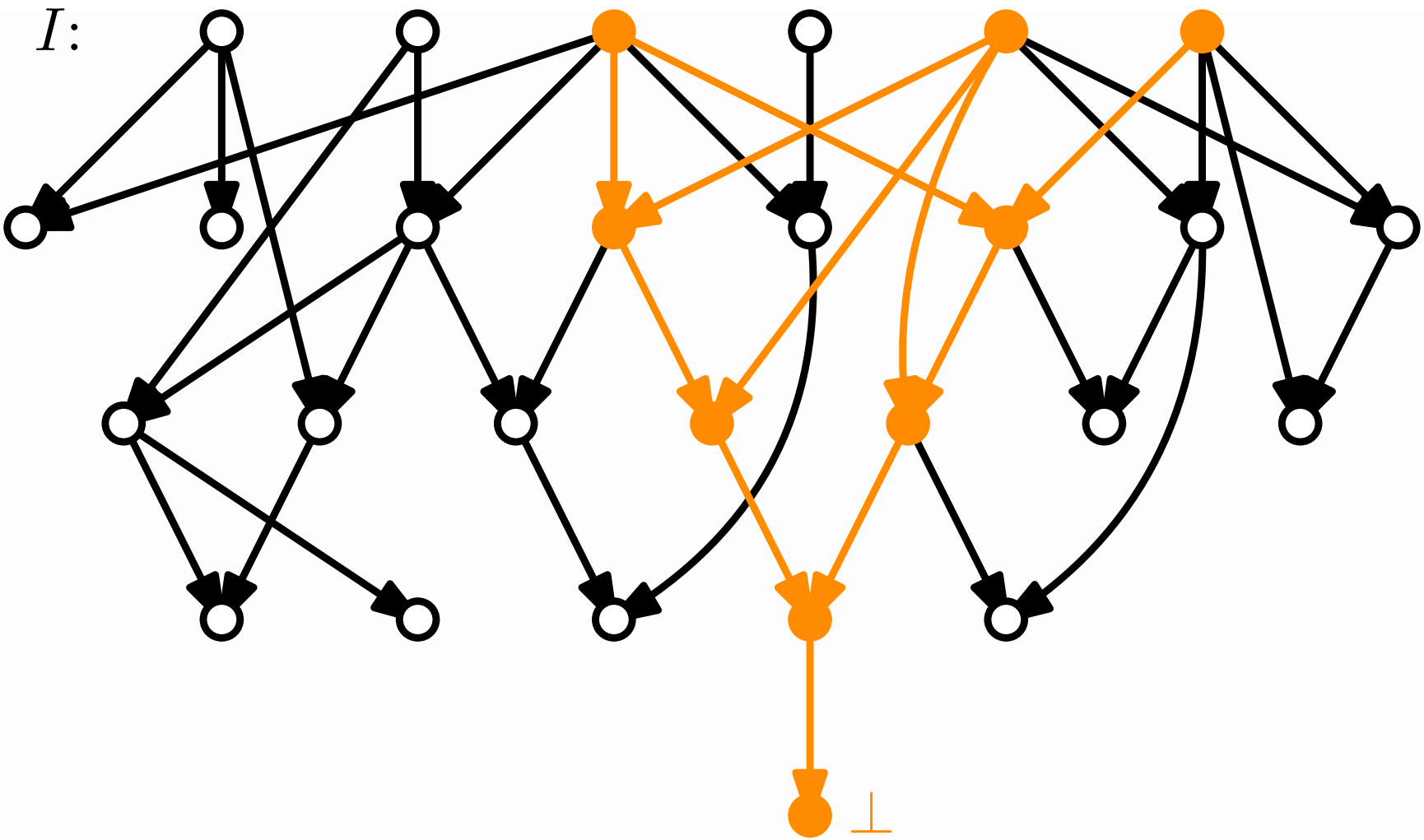

How close can we actually hope get to the perfect clause selection?

How is Clause Selection Traditionally Done?

Take simple clause evaluation criteria:

- *age*: prefer clauses that were generated long time ago
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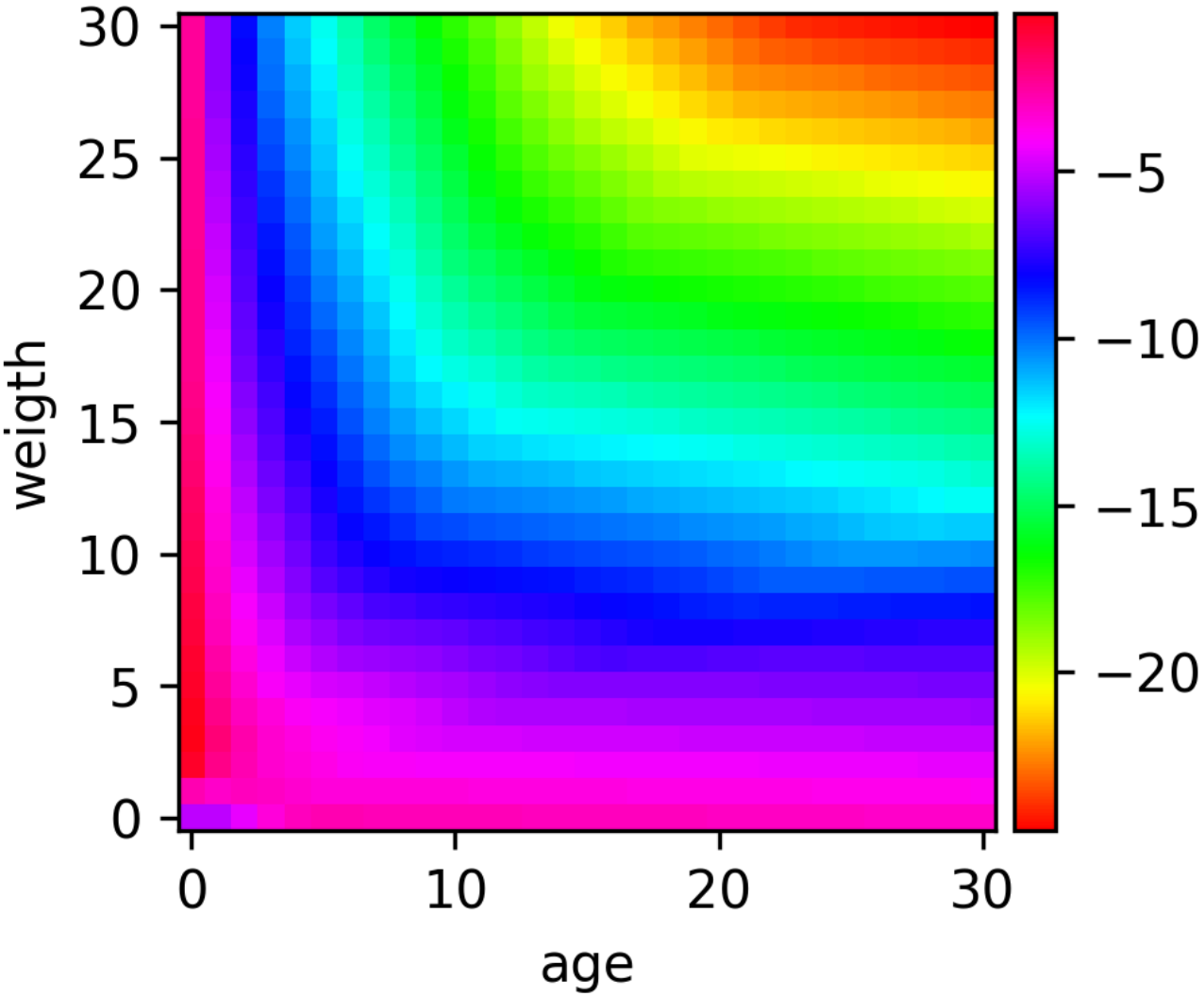
Example (Organizing $Passive$ via two priority queues)

$\mathcal{S}$:

Sneak Peek: What Do NNs Think of Age and Weigth?



Key Reinforcement Learning Concepts

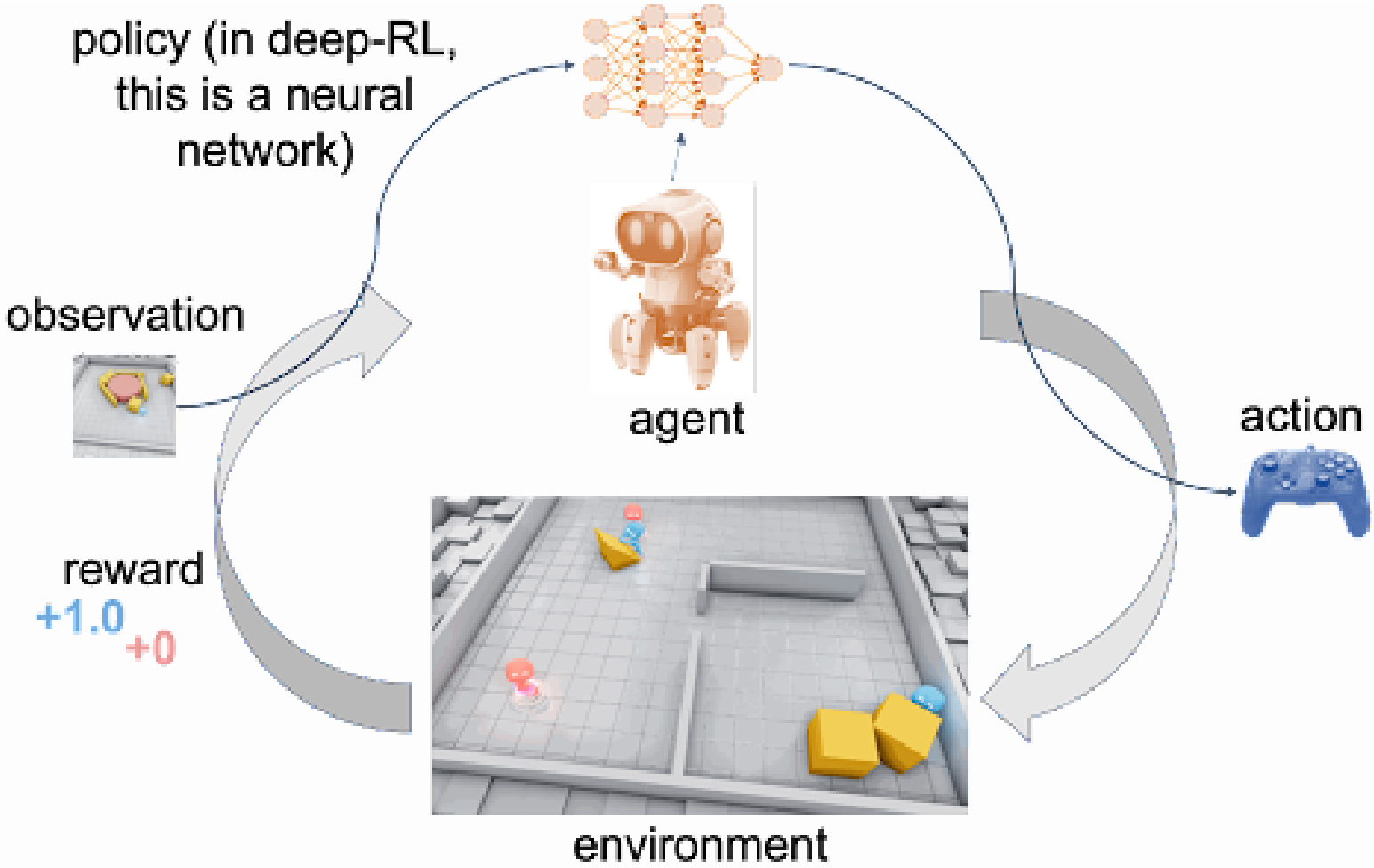


*Illustration from anyscale.com.

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Inspired by the great successes:

- ATARI games (DQN)
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What's really unique about RL?

- It programs itself (sometimes even optimally, in the limit)
- It could discover fundamentally novel tricks and hacks!

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- the clause selection heuristic

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- *static* - the conjecture we are trying to prove
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➥ TRAIL [Crouse et al.'21], [McKeown'23], [Shminke'23], ...

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Reward

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- like in ENIGMA [Jakubův&Urban17]:
  Learn to recognize and prefer for selection clauses that look like those that contributed to a proof in past successful runs.

Towards the RL-Inspired Learning Operator

A *trace* of a successful proof attempt on problem P is a tuple

$$T = (P, \mathcal{C}, \mathcal{C}^+, \{\mathcal{P}_i\}_{i \in I_T}).$$

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- input: neural network $N_{\boldsymbol{\theta}}$ (learnable params $\boldsymbol{\theta}$), set of traces $\mathcal{T}$
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$$\pi_{C,\boldsymbol{\theta}} = \mathsf{softmax}_C\big(\{l_D\}_{D \in \mathcal{P}}\big) = \frac{e^{l_C}}{\sum_{D \in \mathcal{P}} e^{l_D}}$$

is the (stochastic) clause selection policy defined by $N_{\boldsymbol{\theta}}$

The RL-Inspired Operator

Policy Gradient Theorem [Williams'92]

To improve a policy in terms of the expected return we update

$$\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha r_C \nabla_{\boldsymbol{\theta}} \log \pi_{C,\boldsymbol{\theta}},$$

where r_C is the return / reward at the corresponding step.

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Our Operator:

Each moment in time i is an independent opportunity to improve, with

$$\delta_i^T = \text{mean}_{C \in \mathcal{P}_i^+} \nabla_{\boldsymbol{\theta}} \log \pi_{C,\boldsymbol{\theta}},$$

for a trace $T = (P, \mathcal{C}, \mathcal{C}^+, \{\mathcal{P}_i\}_{i \in I_T})$ and $\mathcal{P}_i^+ = \mathcal{P}_i \cap \mathcal{C}^+$. Then

$$\delta^T = \text{mean}_{i \in I_T} \delta_i^T \text{ and } \delta = \text{mean}_{T \in \mathcal{T}} \delta^T.$$

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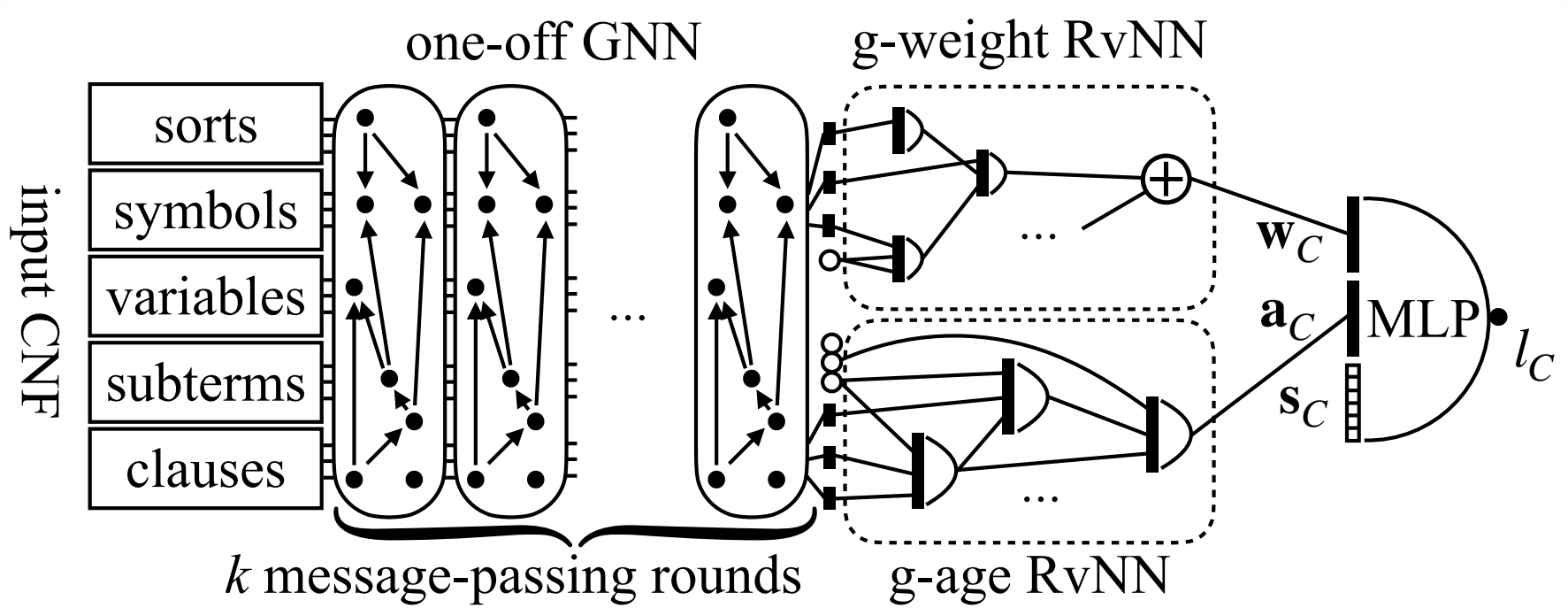
Simple Hand-Crafted Features on Top!

Architecture Diagram



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Single Clause Queue:

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Iterative Improvement Loop:

- run (guided/plain) prover, collect traces, train from traces
- repeat

Experiments

Setup:

- TPTP v9 CNF+FOF, $19\,477$ problems (train/test split)
- Vampire's default strategy (1:1 age-weight alternation)
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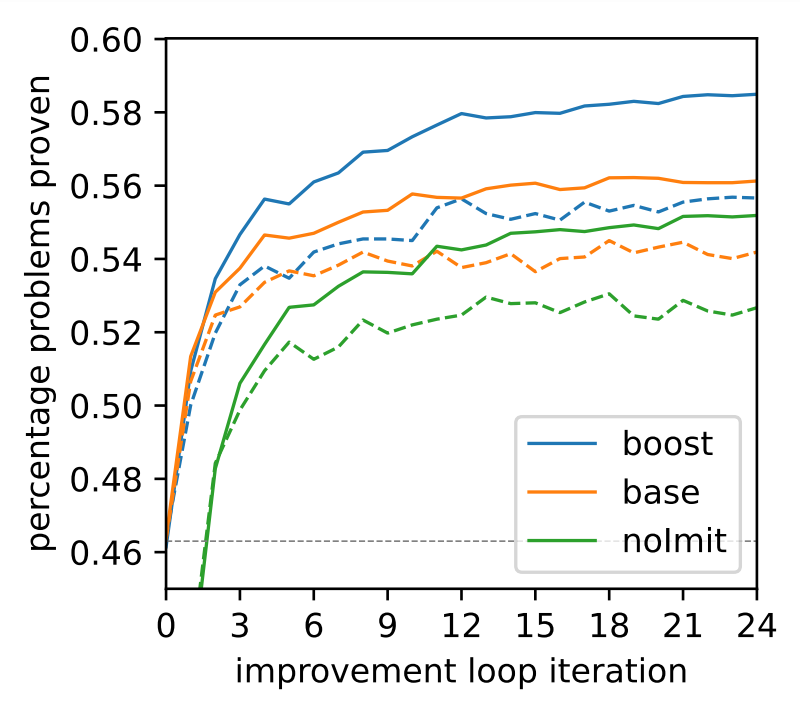

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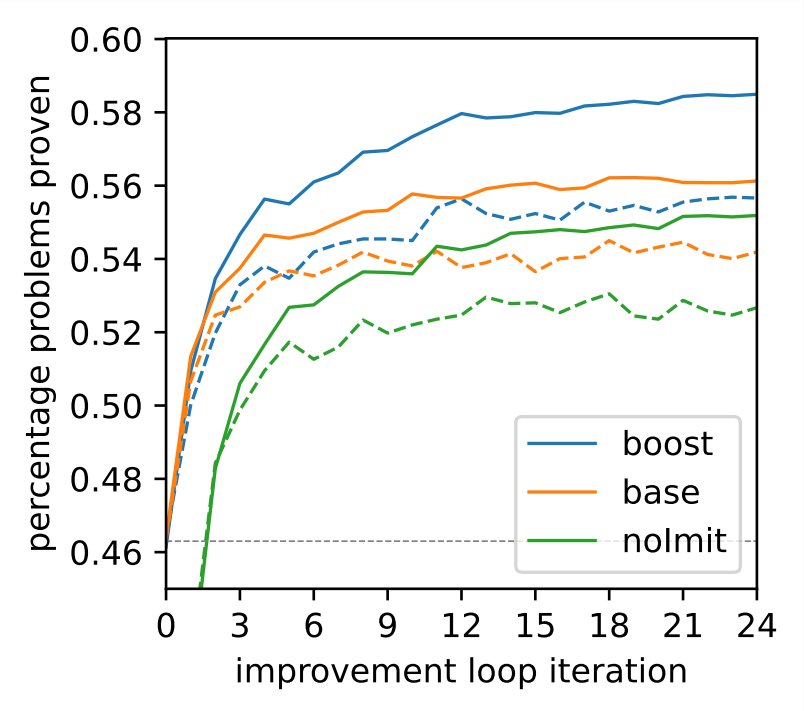

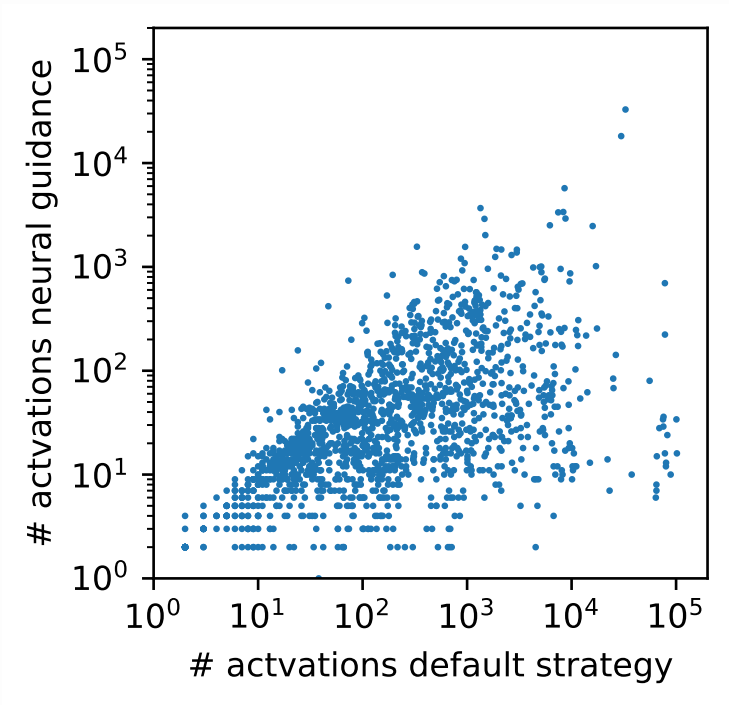

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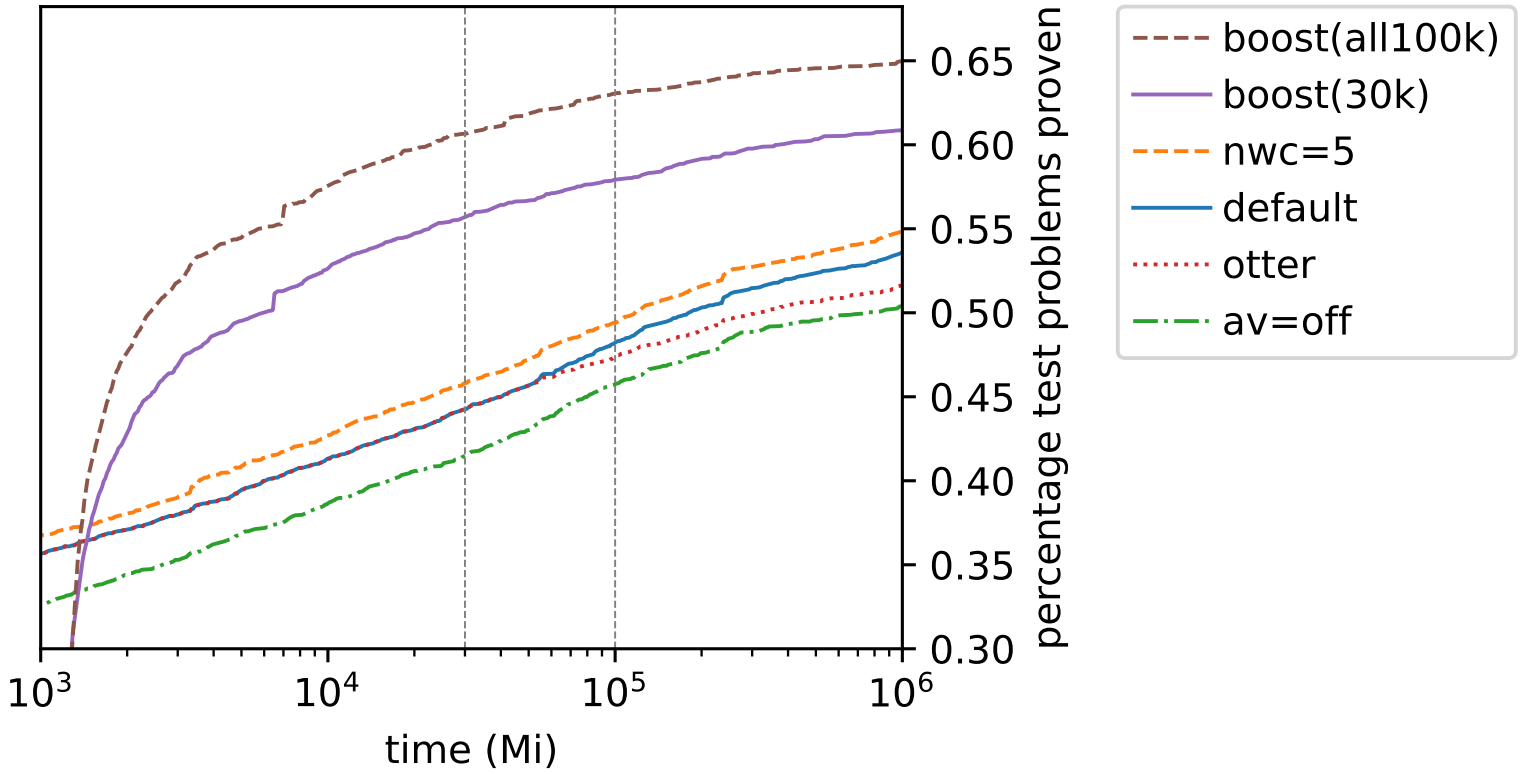

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Train the network from previous runs

- What lead to a success? / What was a reason for failure?
- The proxy (ML) task vs the true target

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Intuition: A first order variable (such as X) stands for all possible objects of the domain (recall the implicit $\forall$ in front of each clause).

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$$\frac{L_1 \vee C_1 \quad \neg L_2 \vee C_2}{(C_1 \vee C_2)\sigma} \text{ (BR).}$$

where σ is the most general unifier of L_1 and L_2. (Fact analogously).

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Here: $\sigma = \{X/\mathrm{socrates}\}$

A Bit Less Trivial Example(s)

Are the following two atoms unifiable?

$$p(Z, f(X, b, Z)) \qquad p(h(X), f(g(a), Y, Z))$$

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And what about these?

$$p(g(X, a, Z), Z) \qquad p(g(k(Z), Y, Z), h(X))$$

Some Terminology First

Recall: Terms

- every variable $x \in \mathcal{V}$ is a term
- if $f/k \in \Sigma_F$ and $t_1, \ldots, t_k$ are terms, then $f(t_1, \ldots, t_k)$ is a term

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- A term t is called ground if $Var(t) = \emptyset$.

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Notation and conventions

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Variables $x_1, \ldots, x_n$ are said to be bound to the terms $t_1, \ldots, t_n$, resp.
A pair x_i/t_i is called a binding.

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- $Dom(\theta) = \{x_1, \ldots, x_n\}$, $Rng(\theta) = \{t_1, \ldots, t_n\}$

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Consider a substitution $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$

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Example (Why "simultaneous"?)

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$$g(x, y)\theta = g(f(y), c).$$

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Lemma (Variant)

A term t is a variant of s iff t is an instance of s and s an instance of t.

Composition

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Lemma (Composition)

Let $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$ and $\eta = \{y_1/s_1, \ldots, y_m/s_m\}$. Then $\theta\eta$ can be constructed from the sequence

$$x_1/t_1\eta, \ldots, x_n/t_n\eta, y_1/s_1, \ldots, y_m/s_m$$

1. by removing the bindings $x_i/t_i\eta$ for which $x_i = t_i\eta$ and
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For $\theta = \{u/z, x/3, y/f(x, 1)\}$ and $\eta = \{x/4, z/u\}$ we get

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  (Assume $\{x/a\} = \{x/y\}\eta$ for some η. Since $x/a \in \tau = \{x/y\}\eta$, we have $y/a \in \eta$, which implies $y \in Dom(\tau)$. A contradition.)

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Recall the convention about symbols and variables. Here we had: $\Sigma_f = \{a/0, b/0, f/1, g/2\}$ (constants and function symbols) and $\mathcal{V} = \{x, y, z, \ldots\}$ (first-order variables).

Unifying Sets of Pairs of Terms

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Lemma (Solved Form)

If $E = \{x_1 \doteq t_1, \ldots, x_n \doteq t_n\}$ is solved, then $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$ is an mgu of E. **Proof:**

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  Thus $\eta = \theta\eta$. (Such an mgu is, by the way, called strong.)

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(We call: 2. the clash-failure and 6. the "occurs check"-failure.)

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If the original set E has a unifier, MM successfully terminates with an equivalent solved set E'. Otherwise, it terminates with failure.

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$$f(x_1, x_2, x_3) \doteq f(g(x_0, x_0), g(x_1, x_1), g(x_2, x_2))$$
$$\theta_3 = \theta_2 \cup \{x_3 / g(g(g(x_0, x_0), g(x_0, x_0)), g(g(x_0, x_0), g(x_0, x_0)))\}$$
$$\vdots$$

A First-order Resolution Calculus in Full

Consists of two inference rules:

- Binary resolution, denoted by BR:

$$\frac{L_1 \vee C_1 \quad \neg L_2 \vee C_2}{(C_1 \vee C_2)\sigma} \text{ (BR)}.$$

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